Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

AN APPLICATION OF FOURIER'S SERIES TO PROBABILITY.

BY DAVID F. BARROW.

Introduction.

Many persons have been interested in compounding frequencies. They all seem to aim at asymptotic expressions which represent approximately the functions obtained by compounding a large number of frequencies. The most general theorem of this sort is to the effect that if a great many frequencies be compounded the result approaches a Gaussian error function.* The generality and beauty of this theorem, and the fact that in most applications a great many frequencies enter, seem to have led writers to give little attention to those cases where only a few frequencies enter. Furthermore in those cases where exact formulas for compounding a few frequencies have been given, great difficulty is found in obtaining numerical results. Even Laplace confesses that in one case such a difficulty arrested him for a long time before he found a suitable approximation.

This article gives a simple method of compounding frequency functions when they can be represented by Fourier's series. It has the advantages of being applicable over a wide range of problems and of yielding formulas which represent exactly the frequencies and from which numerical results can be obtained with relative ease. Owing to the existence of asymptotic formulas, it is most useful where a few frequencies are to be compounded, that is where the asymptotic solution is not very exact.

In section 1 the general method is set forth. In section 2 a particular problem is solved. The solution looks so unlike the asymptotic solution that it seems of interest to compare the two directly to get an upper limit on the discrepancy. This is done in sections 4 and 5, and also a way is pointed out for comparing the exact and asymptotic solutions in various other cases.

1. Fundamental theorems.

In many probability problems we consider a variable t which may take on any one of an infinite number of values, for instance any value in some interval, finite or infinite. Of course in such a case the probability that any particular value will be assumed is infinitesimal, but it is often possible to define a function $f(x)$, called the frequency of t, such that the integral of $f(x)$ from any value a to any larger value b gives the probability that t will assume a value in the interval (a, b). Evidently $f(x)$ must be everywhere positive or zero, and the integral of $f(x)$ from $-\infty$ to $+\infty$ must be unity since this is the symbol for certainty.

---

* For an exact statement and a simple proof of this theorem see Crofton, Phil. Trans., vol. 160, p. 175.

LEMMA. *If $f(x)$ is the frequency of t, and $\varphi(x)$ is the frequency of τ, and $F(x)$ is the frequency of $t + \tau$; then*

$$F(x) = \int_{-\infty}^{\infty} f(x - \lambda)\varphi(\lambda)d\lambda = \int_{-\infty}^{\infty} f(\mu)\varphi(x - \mu)d\mu.$$

Proof. The probability that $t + \tau$ will fall between x and $x + \Delta x$ may be expressed in two ways. First it is by definition the integral of $F(\lambda)$ from x to $x + \Delta x$, and secondly it is the double integral of $f(\mu)\varphi(\lambda)$ extended over that region of the λ, μ-plane which lies between the two lines $\lambda + \mu = x$ and $\lambda + \mu = x + \Delta x$. Equating these two expressions we have

$$\int_{x}^{x+\Delta x} F(\lambda)d\lambda = \int_{-\infty}^{\infty}\int_{x-\lambda}^{x+\Delta x-\lambda} \varphi(\lambda)f(\mu)d\mu d\lambda = \int_{-\infty}^{\infty}\int_{x-\mu}^{x+\Delta x-\mu} \varphi(\lambda)f(\mu)d\lambda d\mu.$$

Dividing by Δx and taking the limit for $\Delta x = 0$ we obtain the desired result.

THEOREM 1. *If*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{i=1}^{\infty}\left(a_i \cos\frac{i\pi x}{C} + b_i \sin\frac{i\pi x}{C}\right)$$

and

$$\varphi(x) = \tfrac{1}{2}\alpha_0 + \sum_{i=1}^{\infty}\left(\alpha_i \cos\frac{i\pi x}{C} + \beta_i \sin\frac{i\pi x}{C}\right)$$

and

$$F(x) = \tfrac{1}{2}A_0 + \sum_{i=1}^{\infty}\left(A_i \cos\frac{i\pi x}{C} + B_i \sin\frac{i\pi x}{C}\right)$$

represent the frequencies of t, τ, and $t + \tau$ respectively in the interval $(-C, C)$, while outside this interval all three functions are identically zero; then

$$A_0 = Ca_0\alpha_0,$$

$$A_i = C(a_i\alpha_i - b_i\beta_i),$$

$$B_i = C(a_i\beta_i + b_i\alpha_i) \qquad (i = 1, 2, 3, \cdots).$$

Proof. In applying the lemma to calculate $F(x)$ from $f(x)$ and $\varphi(x)$, the limits of integration may be reduced to $-C$ and C, since outside this interval the integrand is identically zero. Now

$$f(x-\lambda) = \tfrac{1}{2}a_0 + \sum_{i=1}^{\infty}\left[a_i \cos\frac{i\pi}{C}(x-\lambda) + b_i \sin\frac{i\pi}{C}(x-\lambda)\right]$$

$$= \tfrac{1}{2}a_0 + \sum_{i=1}^{\infty}\left[a_i \cos\frac{i\pi x}{C}\cos\frac{i\pi\lambda}{C} + a_i \sin\frac{i\pi x}{C}\sin\frac{i\pi\lambda}{C}\right.$$

$$\left. + b_i \sin\frac{i\pi x}{C}\cos\frac{i\pi\lambda}{C} - b_i \cos\frac{i\pi x}{C}\sin\frac{i\pi\lambda}{C}\right].$$

To obtain $F(x)$ we multiply this series, term by term, by the series representing $\varphi(\lambda)$, and integrate with respect to λ from $-C$ to C.* On account of the orthogonal property of the sines and cosines involved a great many of the resulting terms are zero. The coefficients of the remaining terms give us the above values for A and B in terms of a, b, α and β.

We have slurred over one difficulty, namely that when we represent the function $f(x-\lambda)$ by the Fourier's series where x and λ can each range over the interval $(-C, C)$, the value of the argument $x-\lambda$ may lie outside this interval and for such values the series may fail to represent $f(x-\lambda)$ correctly. We shall show that whenever this occurs the series representing $\varphi(\lambda)$ vanishes so that the result of the integration is not vitiated. Let (d_1, d_2) and (δ_1, δ_2) be the smallest intervals outside of which $f(x)$ and $\varphi(x)$ respectively vanish identically. Then the corresponding interval for $F(x)$ will be $(d_1+\delta_1, d_2+\delta_2)$. Now the interval $(-C, C)$ must by hypothesis include these three intervals. Hence the series for $f(x)$ must converge to zero in the two intervals $(-C, d_1)$ and (d_2, C), and since it is periodic it will converge to zero for a distance beyond the ends of $(-C, C)$ and hence will represent $f(x)$ in the wider interval $(-2C+d_2, 2C+d_1)$. We need, therefore, look at only two cases: $x-\lambda < -2C+d_2$, and $x-\lambda > 2C+d_1$. Take the first case, then

$$x-\lambda < -2C+d_2;$$

but

$$-x < C$$

and

$$d_2+\delta_2 < C$$

Adding

$$\delta_2-\lambda < 0 \quad \text{or} \quad \lambda > \delta_2.$$

Hence λ must lie between δ_2 and C, and the Fourier's series for $\varphi(\lambda)$ will converge to zero. We find similarly for the second case that λ must lie between $-C$ and δ_1, so that the series for $\varphi(\lambda)$ is zero in this case also.

---

* The uniform convergence of Fourier's series is sufficient to permit this operation, cf. Picard, Traite d'Analyse, Vol. 1, chapter on *trigonometric series.*

The results of theorem 1 may be put into the following compact form by defining $b_0, = \beta_0 = 0$, and making use of the imaginary unit $\sqrt{-1}$:

$$A_i + B_i\sqrt{-1} = C(a_i + b_i\sqrt{-1})(\alpha_i + \beta_i\sqrt{-1}) \qquad (i = 0, 1, 2, \cdots).$$

Equating real and imaginary parts gives the formulas. From this form it is evident that to find the frequency of the sum of n variables we need only multiply the n parentheses together and prefix the factor C^{n-1}. Then the real and imaginary parts of the product give the coefficients in the Fourier's series for the frequency of the sum.

An alternate form of theorem 1 is

THEOREM 2. *If*

$$f(x) = \tfrac{1}{2}r_0 + \sum_{i=1}^{\infty} r_i \cos\left(\frac{i\pi x}{C} - g_i\right),$$

$$\varphi(x) = \tfrac{1}{2}\rho_0 + \sum_{i=1}^{\infty} \rho_i \cos\left(\frac{i\pi x}{C} - \gamma_i\right),$$

$$F(x) = \tfrac{1}{2}R_0 + \sum_{i=1}^{\infty} R_i \cos\left(\frac{i\pi x}{C} - G_i\right),$$

represent the frequencies of t, τ, and $t + \tau$ respectively in the interval $(-C, C)$ and outside this interval all three functions are zero, then

$$R_i = Cr_i\rho_i \qquad (i = 0, 1, 2, \cdots),$$

$$G_i = g_i + \gamma_i \qquad (i = 1, 2, 3, \cdots).$$

An interesting special case occurs when the three functions are even in addition to satisfying the conditions named in theorem 1, for then each can be expressed in a series of cosines alone, causing obvious simplifications in our formulas.

Not only can theorem 1 be used to find the frequency for the sum of several variables whose frequencies are known, but it may also be used to reverse the process. For example if the frequency for the sum of two variables is given, and that for one of the variables, then the frequency for the other may be found. Again, if an arbitrary frequency function be given, it may be possible in various ways to assign two or more variables with suitable frequencies so that their sum shall have the given frequency. The writer hopes to take up this question later.

2. **Solution of a problem.**

As an application of the general method let us solve the following problem.

"There are n numbers each lying between $-\frac{1}{2}$ and $+\frac{1}{2}$, such that any value of each between these limits is equally probable. What is the probability that their sum will lie between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, $\cdots$?" *

Let $f(x, n)$ denote the frequency of the sum of n such numbers (or variables). Then $f(x, 1)$ is identically zero outside the interval $(-\frac{1}{2}, \frac{1}{2})$, since the variable whose frequency it is cannot occur outside this interval. Furthermore all values between $-\frac{1}{2}$ and $+\frac{1}{2}$ are equally probable, so that the integral of $f(x, 1)$ over any subinterval thereof should be proportional to the length of that subinterval. Hence $f(x, 1)$ must equal a constant in this interval, and this constant must be unity in order that the integral of $f(x, 1)$ from $-\frac{1}{2}$ to $+\frac{1}{2}$ be equal to unity, the symbol for certainty. Thus $f(x, 1)$ is an even function and may be developed into the following series of cosines:

$$f(x, 1) = \frac{1}{2C} + \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\frac{i\pi}{2C}\cos\frac{i\pi x}{C}.$$

The frequency for the sum of n such variables is easily found, by successive applications of Theorem 1, to be

$$f(x, n) = \frac{1}{2C} + \frac{2^n C^{n-1}}{\pi^n}\sum_{i=1}^{\infty}\frac{1}{i^n}\sin^n\left(\frac{i\pi}{2C}\right)\cos\frac{i\pi x}{C}. \tag{1}$$

Obviously if C be taken equal to $n/2$ it will be large enough to include all points where the function does not vanish identically. It may be taken larger without damaging the formula, in which case the series merely converges to zero in the intervals $(n/2, C)$ and $(-C, -(n/2))$. In fact, on account of the periodicity of the cosines, the series converges to zero for a distance $(C - (n/2))$ beyond the ends of the interval $(-C, C)$, and therefore represents correctly $f(x, n)$ when x lies in the interval $((n/2) - 2C, 2C - (n/2))$. Now this last statement holds true even when C has a value between $n/4$ and $n/2$, but the proof is too detailed to be given here. The fact is only of importance in calculations, as we shall wish to take C as small as possible in order to obtain a rapidly converging series. A check calculation may be made by changing the value of C.

Now let $\psi(s, n)$ denote the function which furnishes the solution of our problem. Then

$$\psi(s, n) = \int_{s-\frac{1}{2}}^{s+\frac{1}{2}} f(x, n)dx = \frac{1}{2C} + \frac{2^{n+1}C^n}{\pi^{n+1}}\sum_{i=1}^{\infty}\frac{1}{i^{n+1}}\sin^{n+1}\left(\frac{i\pi}{2C}\right)\cos\frac{i\pi s}{C}.$$

---

* The problem was stated in this form by Prof. E. W. Brown in The American Math. Monthly, Nov., 1916, p. 341. It is a special case of a problem solved by Laplace in his Théorie Analytique des Probabilités (Paris, 1820), pp. 253–256. He finds a formula unsuitable for calculation, and so resorts to an approximate formula which is practically the asymptotic Gaussian error function which we derive in section 4. See collected works of Laplace, Vol. 12, pp. 301 ff.

The series represents $\psi(s, n)$ if $(n + 1)/2 - 2C \leqq s \leqq 2C - (n + 1)/2$. The following table for $n = 100$ was calculated using $C = 40$ and checked (except the last entry) using $C = 30$.

S	0	1	2	3	4	5	6	7	8	9	10
$\psi(S, 100)$....	.13731	.12943	.10841	.08066	.05328	.03124	.01624	.00748	.00305	.00110	.00035

For negative values of s the values of ψ will be the same as for positive values. The calculation is rather tedious as fifteen or twenty terms of the series have to be used. If no great accuracy is required, it is possible to use in this, and, in many other cases, certain well-known asymptotic formulas. They are taken up in the last section.

3. A Fourier's series approximating the Gaussian error curve.

The general theory of compounding frequencies leads us to suspect that a Gaussian error function would give a good approximation to the frequency in the problem of the last section and similar ones. The results of this section are developed chiefly for the purpose of taking this point up in the next section, yet they present some interest of their own. If we develop the function $a/\sqrt{\pi}\cdot e^{-a^2x^2}$ in a Fourier's series in the interval $(-C, C)$, we obtain a series of cosines alone, since the function is even. The ith coefficient is

$$a_i = \frac{2}{C}\int_0^C \frac{a}{\sqrt{\pi}} e^{-a^2\lambda^2} \cos\frac{i\pi\lambda}{C}\,d\lambda.$$

This cannot be integrated in closed form. We therefore adopt an approximation by extending the upper limit to infinity, thereby making an error in each coefficient, the total effect of which is to be examined presently. Denoting by $\bar{a}_i$ this approximate value of the ith coefficient, we have*

$$\bar{a}_i = \frac{2}{C}\int_0^\infty \frac{a}{\sqrt{\pi}} e^{-a^2\lambda^2} \cos\frac{i\pi\lambda}{C}\,d\lambda = \frac{1}{C} e^{-(i^2\pi^2/4a^2C^2)}.$$

Hence the approximate development of $a/\sqrt{\pi}\cdot e^{-a^2x^2}$ is

$$\frac{1}{2C} + \frac{1}{C}\sum_{i=1}^{\infty} e^{-(i^2\pi^2/4a^2C^2)} \cos\frac{i\pi x}{C}. \tag{2}$$

Now let us denote by $\epsilon(x)$ the error made by substituting this series for the Gaussian error function. Then

$$\epsilon(x) = \tfrac{1}{2}\cdot\frac{2}{C}\int_C^\infty \frac{a}{\sqrt{\pi}} e^{-a^2\lambda^2} d\lambda + \sum_{i=1}^{\infty}\left[\frac{2}{C}\int_C^\infty \frac{a}{\sqrt{\pi}} e^{-a^2\lambda^2} \cos\frac{i\pi\lambda}{C}\,d\lambda\right] \cos\frac{i\pi x}{C}$$

---

* For this integration see B. O. Pierce, A Short Table of Integrals, p. 64, Formula 508.

$$= \frac{a}{C\sqrt{\pi}}\left[\int_{C}^{2C} e^{-a^2\lambda^2}d\lambda + \int_{2C}^{3C} e^{-a^2\lambda^2}d\lambda + \cdots\right]$$

$$+ \frac{2a}{C\sqrt{\pi}}\sum_{i=1}^{\infty}\left[\int_{C}^{2C} e^{-a^2\lambda^2}\cos\frac{i\pi\lambda}{C}\,d\lambda + \int_{2C}^{3C} e^{-a^2\lambda^2}\cos\frac{i\pi\lambda}{C}\,d\lambda + \cdots\right]\cos\frac{i\pi x}{C}.$$

Next change the variables of integration as follows: In the first integral of each term of the sum, let $\lambda = 2C - \mu$, in the second integral of each term of the sum, let $\lambda = 2C + \mu$, in the third integral of each term of the sum, let $\lambda = 4C - \mu$, in the fourth integral of each term of the sum, let $\lambda = 4C + \mu$, and so on regularly. This is designed to reduce each set of limits to zero and C without introducing troublesome negative signs. After slight trigonometric reductions and judicious changes of limits we obtain

$$\epsilon(x) = \frac{a}{C\sqrt{\pi}}\int_{0}^{C}\left[e^{-a^2(2C-\mu)^2} + e^{-a^2(2C+\mu)^2} + \cdots\right]d\mu$$

$$+ \frac{2a}{C\sqrt{\pi}}\sum_{i=1}^{\infty}\int_{0}^{C}\left[e^{-a^2(2C-\mu)^2} + e^{-a^2(2C+\mu)^2} + \cdots\right]\cos\frac{i\pi\mu}{C}\,d\mu\cdot\cos\frac{i\pi x}{C}.$$

But we recognize this as the Fourier's series which represents that function of x obtained by substituting x for μ in the bracket. Hence

$$\epsilon(x) = \frac{a}{\sqrt{\pi}}\left[e^{-a^2(2C-x)^2} + e^{-a^2(2C+x)^2} + e^{-a^2(4C-x)^2} + \cdots\right].$$

Thus we have found not merely an upper limit on the error but the exact error for any value of x, given as a function of x. Since x must lie in the interval $(-C, C)$ this error is very small even for moderately large values of a and C. Its greatest value occurs when $x = C$, and its smallest value when $x = 0$. The property most valuable to us in what follows is that for a fixed value of a the error approaches zero if C is allowed to increase indefinitely.

It is rather curious that the Fourier's series (2) is equal to $a/\sqrt{\pi}\cdot e^{-a^2x^2} + \epsilon(x)$ not only in the interval $(-C, C)$ but for all values of x, since the latter function is periodic with a period of $2C$.

4. Asymptotic formulas for some frequencies.

Let us compare coefficients in series (1) and (2) for the purpose of determining a value of a which shall make the Gaussian error function approximately represent the frequency in the problem of section 2. For this we shall use the following more or less obvious formulas:

$$e^{-(y^2/6)} = 1 - \frac{y^2}{6} + \frac{y^4}{72} - \cdots,$$

$$\frac{\sin y}{y} = 1 - \frac{y^2}{6} + \frac{y^4}{120} - \cdots,$$

$$(e^{-(y^2/6)})^n - \left(\frac{\sin y}{y}\right)^n = \left(e^{-(y^2/6)} - \frac{\sin y}{y}\right)\sum_{i=0}^{n-1}(e^{-(y^2/6)})^i\left(\frac{\sin y}{y}\right)^{n-i-1},$$

$$(3) \qquad (e^{-(y^2/6)})^n - \left(\frac{\sin y}{y}\right)^n \leqq \frac{ny^4}{72}(e^{-(y^2/6)})^{n-1} \leqq \frac{27n}{100(n-1)^2}.$$

The last inequality is obtained by taking the maximum of the preceding function of y.

Now write down the difference between the ith coefficients in series (2) and (1), changing their forms slightly to facilitate comparison with formula (3). This gives

$$\frac{1}{C}\left[e^{-1/6(i\pi/2C)^2}\right]^{6/a^2} - \frac{1}{C}\left[\frac{\sin\left(\frac{i\pi}{2C}\right)}{\left(\frac{i\pi}{2C}\right)}\right]^n.$$

If we take a to satisfy $6/a^2 = n$, or $a = \sqrt{6/n}$, then supposing n to be rather large, say twenty or more, we can simplify the maximum difference given by (3) and say that the maximum difference between any two corresponding coefficients in series (2) and (1) does not exceed $3/10\,Cn$.

Next let k denote the smallest integer greater than or equal to $2C$ and consider the remainder of series (1) after the kth term. This remainder is obviously less than the following

$$\frac{2^nC^{n-1}}{\pi^n}\sum_{i=k+1}^{\infty}\frac{1}{i^n} < \frac{2^nC^{n-1}}{\pi^n}\int_{2C}^{\infty}\frac{d\lambda}{\lambda^n} = \frac{2}{(n-1)\pi^n}.$$

The remainder of series (2) after k terms is still smaller. Now the average difference between corresponding coefficients in the first k terms in series (2) and (1) will not exceed one half the maximum, as is seen from (3), so that in the sum of these terms we cannot have an error greater than $3/10\,n$. This limit is extravagant enough to cover the error made in neglecting terms beyond the kth. Also the error discussed in section (3) can be made as small as we please by taking C large. Hence *if n is not less than twenty in the problem of section* (2), *the Gaussian error function with $a = \sqrt{6/n}$ approximates the frequency, the discrepancy being less than* $3/10\,n$. In actual computation the discrepancy seems to be much less. For the case $n = 100$ it is about $1/100\,n$.

As n grows infinite every ordinate in the graphs of both the frequency function and the Gaussian function approaches zero, so that it is evident a priori that the difference between corresponding ordinates must approach zero. This might seem to rob our result of all significance; but it will be noticed that the greatest ordinate in the Gaussian curve is $\sqrt{6/n\pi}$ so that the difference between corresponding ordinates near the origin is an infinitesimal of higher order than either of the ordinates, as n grows infinite.

Let us take up another problem, an extension of the previous. Given n variables each lying between $-\frac{1}{2}l$ and $+\frac{1}{2}l$, such that every value of each in this interval is equally probable. Find a Gaussian error function approximating the frequency of their sum. A similar course of reasoning shows that the Gaussian function with $a = \sqrt{6/nl^2}$ gives an approximation to the frequency whose greatest discrepancy is less than $3/10\,nl$. Let us vary the problem by supposing n to increase and l to diminish in such a way that nl^2 remains equal to some arbitrary constant. The approximating Gaussian function would not change, but the discrepancy would approach zero. Evidently a problem of this sort could be framed whose frequency curve closely resembled any arbitrary Gaussian error curve.

Next we shall consider the sum of n variables each having the same frequency $f(x)$, which shall be identically zero outside a certain interval $(-l, l)$, and capable of development in a Fourier's series of cosines in any larger interval $(-C, C)$. Then the frequency of the sum is approximated by the Gaussian error function in which a is determined by

$$\frac{1}{a^2} = 4n \int_0^l \lambda^2 f(\lambda)\,d\lambda, \tag{4}$$

and the approximation becomes better as n increases.

A sketch* of the proof is as follows: We have

$$f(x) = \frac{1}{2C} + \sum_{i=1}^{\infty} a_i \cos\frac{i\pi x}{C}.$$

The constant term is known since $f(x)$ is a frequency function and therefore its integral from $-C$ to C is unity. The frequency for the sum of the n variables is, by application of Theorem 1,

$$\frac{1}{2C} + C^{n-1} \sum_{i=1}^{\infty} a_i^n \cos\frac{i\pi x}{C}. \tag{5}$$

---

[*] Bessel has obtained this result and the following one; Astr. Nachr., vol. 15, pp. 369–405. Our only excuse for giving them here is that the use of Fourier's series in this connection seems new and interesting.

Now we know that a_i is less than A/i, where A is some constant.* Consequently the remainder beyond a certain term in series (5) will be very small. To show that the earlier terms in series (5) and (2) are nearly equal when a is chosen as above, we exhibit a_i as follows:

$$a_i = \frac{2}{C}\int_0^C f(\lambda)\cos\frac{i\pi\lambda}{C}\,d\lambda = \frac{2}{C}\int_0^l f(\lambda)\cos\frac{i\pi\lambda}{C}\,d\lambda$$

$$= \frac{2}{C}\left[\frac{1}{2} - \frac{i^2\pi^2}{2!\,C^2}\int_0^l \lambda^2 f(\lambda)d\lambda + \frac{i^4\pi^4}{4!\,C^4}\int_0^l \lambda^4 f(\lambda)d\lambda - \cdots\right].$$

The last form is obtained by developing $\cos i\pi\lambda/C$ into a power series. Now if we develop the ith coefficient in (2) into a series of powers of the exponent of e, and compare with the above, it will be seen that we have so chosen a in (4) that the coefficients of (5) and (2) are nearly equal for small values of $i\pi/C$, that is for the earlier terms of the two series. The proof can be made rigorous.

A further generalization of the problem is to consider n variables each with a different frequency function, $f_1(x)$, $f_2(x)$, $\cdots$, $f_n(x)$. If each of these functions is zero outside of an interval $(-l, l)$ and capable of development in a Fourier's series of cosines, then the Gaussian error function in which a is determined by

$$\frac{1}{a^2} = 4\sum_{i=1}^{n}\int_0^l \lambda^2 f_i(\lambda)d\lambda$$

will approximate the frequency in an asymptotic manner.

The question of asymptotic formulas when frequencies are compounded which are not *even* functions has been investigated by other methods. It looks as if theorem 1 might be used to establish these results also, but the process would probably be too tedious for interest.

SHEFFIELD SCIENTIFIC SCHOOL,
February 2, 1917.

---

* Picard, loc. cit., p. 253.